SOME INVESTIGATIONS ON FUZZY RELATIONAL OPTIMIZATION MODELS AND THEIR POSSIBLE SOLUTIONS

Synopsis

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Submitted by

Vijay Lakshmi Tiwari

Dr. Antika Thapar
Supervisor
Department of Mathematics

Dr. Richa Bansal
Co-Supervisor
Department of Mathematics

Prof. Ravinder Kumar
Head, Department of Mathematics

Prof. G. S. Tyagi
Dean, Faculty of Science

Department of Mathematics
Faculty of Science
Dayalbagh Educational Institute (Deemed University)
Dayalbagh, Agra-282005
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1. Background and motivation

In real life, one is commonly faced with many classes of non-random objects. If we consider, for instance, such categories as tall man, high pressure, young child, etc. all of them convey a semantic meaning, significant for a certain community; however, the concept of membership of an object to such a class is not obvious. Therefore, application of methods based on formal two-valued logics to describe real world facts may be crucially limited or at least viewed as artificial. The main motive of the fuzzy logic is to capture the vagueness of the real world situations and to express it with appropriate mathematical tools.

A classical set A over universal set X can be represented by a characteristic function $\mu_A : X \to \{0,1\}$, defined as $\mu_A(x) = 1$ if $x \in A$ and $\mu_A(x) = 0$ if $x \notin A$. In 1965, Prof. Zadeh generalized this binary representation of values to a continuous interval by introducing fuzzy sets. A fuzzy set A over universal set X is defined by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$ denotes the membership grade of $x \in X$ in fuzzy set A. In this type of representation an element $x \in X$ is partially in A. Zadeh's contribution of fuzzy sets made a tremendous effect to modern science and engineering.

As fuzzy sets can be viewed as an extension of classical sets; similarly, the concept of a classical relation can be extended to a fuzzy relation. A classical relation R on sets X and Y is a subset of $X \times Y$ defined by characteristic function $R : X \times Y \to \{0,1\}$, consisting of collection of pairs (x, y) where $x \in X$, $y \in Y$ and xRy. On the other hand, a fuzzy relation R on sets X and Y is a fuzzy set on $X \times Y$ defined by the membership function $R : X \times Y \to [0,1]$ where the degree of association or membership grade is also available as an additional information.

Notion of fuzzy relation equations (FRE) was first given by Sanchez [51] in 1976. In many methods of describing a system it is assumed that there exists a functional relation between the input and the output variables of the system. Statistical methods are then widely used for determining an acceptable mathematical model of the process. The problems without fuzzy information can be handled using traditional/classical methods for solving linear and nonlinear programming problems. In many ill-defined processes, decision algorithms may be set up based on numerical and non-numerical (linguistic) kinds of information. One way of modelling such fuzzy systems is by means of FRE. The solution set of FRE is, in general, a non-convex set, therefore traditional/classical methods fail in case of fuzzy linear and nonlinear programming problems. In fact, the idea of FRE used as a model of a system could be useful for a wide class of problems where we are faced with fuzzy information, i.e. medical, economical, social or industrial, image processing etc. Other way of modelling ill-defined processes includes fuzzy expert systems, fuzzy inference theory, fuzzy multi-conditional approximate reasoning, etc. These methods are applicable to the problem they have been designed for and optimization problem cannot be directly associated with these methods.

Resolution problem of FRE is one of the most important and widely studied problems in the field of fuzzy sets and systems. The first step for the resolution of FRE is to establish the

existence of the solution. The solution of FRE lies on a lattice structure. The solution set of a consistent system can be characterized by unique maximum solution and finitely many minimal solutions, or dually, by a unique minimum solution and finitely many maximal solutions. If the solution set is empty, approximate solutions can be found. It has been observed in the literature that the solvability and the solution set of FRE changes according to the selection of compositions, t-norms and the structure of lattice.

It has been observed that finding a simple solving algorithm for the resolution of FRE is still a challenge today. The problem of solving a fuzzy optimization problem is a NP-hard problem in terms of computational complexity. Although there is no polynomial time algorithm to completely solve the FRE unless $P = NP$, a universal algorithm, nevertheless, is still desirable. In this situation, a universal algorithm with effectively less computational complexity is very useful for solving linear or nonlinear fuzzy optimization problems. Moreover, advanced soft computing techniques are needed to keep pace with the development in this field. This motivation forms the basis of research.

2. An introduction to fuzzy relation equations

Significance of the emergence of fuzzy set theory, introduction to fuzzy sets, operations on fuzzy sets, fuzzy arithmetic, fuzzy relations, FRE, fuzzy logic, etc. can be found in Klir & Yuan [22] and Ross [50] in detail.

Let $X \in \Im(P \times Q)$, $A \in \Im(Q \times R)$ and $B \in \Im(P \times R)$ be a fuzzy relations on universal sets $P \times Q$, $Q \times R$ and $P \times R$ of order $s \times m$, $m \times n$ and $s \times n$ respectively. Assume that these three relations constrain each other in such a way that

$$X \circ A = B \tag{2.1}$$

where $\circ$ denotes the max-min composition. This means that

$$\max_{i \in I} \min(x_{ki}, a_{ij}) = b_{kj}, \forall\, k \in K, \forall\, j \in J \tag{2.2}$$

$I = \{1, 2, \ldots, m\}$, $J = \{1, 2, \ldots, n\}$ and $K = \{1, 2, \ldots, s\}$ are the index sets. This means that the matrix equation (2.1) comprises of $s \times n$ simultaneous equations of the form (2.2). When two of the components in each of the equations are given and one is unknown, these equations are referred to as FRE.

When matrices X and A are given and matrix B is to be determined from (2.1), the problem is trivial. It is solved simply by performing the max-min multiplication like operation on X and A as defined by (2.2). Clearly, the solution in this case exists and is unique. The problem becomes far from trivial when one of the two matrices on the left hand side of (2.1) is unknown. In this case the solution is guaranteed neither to exist nor to be unique.

Since B in (2.1) is obtained by composing X and A, it is suggestive to view the problem of determining X (or, alternatively, A) from B and A (or, alternatively, B and X) as a decomposition of B with respect to A (or, alternatively with respect to X). Since many problems in various contexts can be formulated as problem of decomposition, the utility of any method for solving (2.1) is quite high.

Suppose that there is a method for solving (2.1) only for the first decomposition problem (given A and B). Then we can indirectly utilize this method for solving second decomposition problem as well. We rewrite (2.1) in the form $A^{-1} \circ X^{-1} = B^{-1}$. Employing transposed matrices, this equation is solvable for A^{-1} by the given method and, then, obtain the solution of (2.1) by $(A^{-1})^{-1} = A$. It is easy to see that the problem (2.1) can be decomposed into a set of simpler problems expressed by the matrix equations

$$x_k \circ A = b_k \tag{2.3}$$

for all $k \in K$ where $x_k = [x_{ki} \mid i \in I]$ and $b_k = [b_{kj} \mid j \in J]$. Since (2.3) is simpler form of (2.1), we need only methods for solving equations of the form as in (2.3) in order to arrive at a solution. Moreover, system of FRE (2.3) can be further simplified as

$$x \circ A = b \tag{2.4}$$

where $x = [x_i \mid i \in I]$, $A = [a_{ij} \mid i \in I, j \in J]$ and $b = [b_j \mid j \in J]$. The resolution problem of FRE is to determine a vector $x = [x_i]_{i \in I}, 0 \le x_i \le 1$ such that (2.4) holds.

2.1. System of sup-$\otimes$ fuzzy relation equations

Let $A = [a_{ij}]$, $0 \le a_{ij} \le 1$, be a $m \times n$ dimensional matrix, $b = [b_j]$, $0 \le b_j \le 1$, be a n-dimensional vector and $I = \{1, 2, \ldots, m\}$, $J = \{1, 2, \ldots, n\}$ be the index sets. Then, the fuzzy relation equations with sup-$\otimes$ composition is given as,

$$x \circ A = b \tag{2.5}$$

where $\circ$ denotes the sup-$\otimes$ composition of x and A, $\otimes$ being a continuous t-norm, i.e.

$$\max_{i \in I} (x_i \otimes a_{ij}) = b_j, \forall\ j \in J \tag{2.6}$$

The resolution problem of FRE (2.5) to determine a vector $x = [x_i]_{i \in I}, 0 \le x_i \le 1$ such that (2.6) holds.

Let $X(A,b) = \{x \in [0,1]^m \mid x \circ A = b\}$ be the solution set of FRE (2.5). The system $x \circ A = b$ is called consistent if $X(A,b) \neq \phi$. Otherwise, it is inconsistent. For any $x^1, x^2 \in X$, we say $x^1 \le x^2$ if and only if $x_i^1 \le x_i^2, \forall\, i \in I$. Therefore $\le$ forms a partial ordering relation on X and $(X, \le)$ becomes a lattice. Equations in (2.5) form a system of latticized polynomial equations. $\hat{x} \in X(A,b)$ is the maximum solution, if $x \le \hat{x}, \forall\, x \in X(A,b)$. Similarly, $\check{x} \in X(A,b)$ is a minimal solution, if $x \le \check{x}$ implies $x = \check{x}, \forall\, x \in X(A,b)$. According to [22], if $X(A,b) \neq \phi$, then it is, in general, a non-convex set which can be completely determined by unique maximum solution $\hat{x}$ and several minimal solutions $\check{x}$.
The maximum solution can be computed explicitly by the residual implicator (pseudo complement). The maximum solution of FRE (2.5) can be obtained by assigning

$$\hat{x} = A \to b = \left[\min_{j \in J} (a_{ij} \to b_j) \right]_{i \in I} \tag{2.7}$$

where

$$a_{ij} \to b_j = \sup \{ x_i \in [0,1] \mid (x_i \otimes a_{ij}) \le b_j \} \tag{2.8}$$

If $\breve{X}(A,b)$ denotes the set of all minimal solutions, then the complete solution set of FRE (2.5) can be formed as (2.9).

$$X(A,b) = \bigcup_{\breve{x} \in \breve{X}(A,b)} \{ x \in [0,1]^m \mid \breve{x} \le x \le \hat{x} \} \tag{2.9}$$

2.2. System of interval valued sup-$\otimes$ fuzzy relation equations

Let $A^- = [a_{ij}^-]$, $A^+ = [a_{ij}^+]$, with $0 \le a_{ij}^- \le a_{ij}^+ \le 1$, be $m \times n$ dimensional matrices, $b^- = [b_j^-]$, $b^+ = [b_j^+]$, with $0 \le b_j^- \le b_j^+ \le 1$, be n-dimensional vectors. Then the system of interval valued FRE with sup-$\otimes$ composition, $\otimes$ being a continuous t-norm is given as

$$b_j^- \le \max_{i=1}^{m} (x_i \otimes a_{ij}) \le b_j^+, \forall\, a_{ij} \in [a_{ij}^-, a_{ij}^+] \tag{2.10}$$

The resolution problem of FRE (2.10) is to determine a vector $x = [x_i]_{i \in I}, 0 \le x_i \le 1$ such that (2.10) holds. For $A^- = A^+$ and $b^- = b^+$, problem (2.10) reduces to (2.6). The solution set of this problem can be completely determined by one maximum solution and a finite number of minimal solutions when it is not empty. The maximum solution can be computed explicitly by the residual implicator (pseudo complement). The maximum solution of fuzzy relation equations (2.10) can be obtained by assigning

$$\hat{x} = A \to b = \left[\min_{j \in J} (a_{ij}^+ \to b_j^+) \right]_{i \in I} \tag{2.11}$$

where

$$a_{ij}^+ \to b_j^+ = \sup \{ x_i \in [0,1] \mid (x_i \otimes a_{ij}^+) \le b_j^+ \} \tag{2.12}$$

2.3. System of inf-$\to$ fuzzy relation equations

In [51], Sanchez suggested the dual composition expressed by inf-$\to$ operation. Necessary and sufficient condition of solvability, characterization of a set of all solutions of this composition has been investigated in [3,43].

Let $A = [a_{ij}]$, $0 \le a_{ij} \le 1$, be a $m \times n$ dimensional matrix, $b = [b_j]$, $0 \le b_j \le 1$, be a n-dimensional vector and $I = \{1, 2, \ldots, m\}$, $J = \{1, 2, \ldots, n\}$ be the index sets. Then, the fuzzy relation equations with inf-$\to$ composition is given as,

$$x \circ^{\rightarrow} A = b \tag{2.13}$$

where $\circ^{\rightarrow}$ denotes the inf-$\rightarrow$ composition of x and A, $\rightarrow$ being a residual implicator, i.e.

$$\inf_{i \in I} (x_i \rightarrow a_{ij}) = b_j, \forall\ j \in J \tag{2.14}$$

Given A and b, the resolution of (2.13) means to find a vector $x = [x_i]_{i \in I}, 0 \le x_i \le 1$ such that (2.13) holds. The maximum solution of this problem is given as $\hat{x} = b \circ^{\rightarrow} A^{-1}$. But no well known method is prescribed in literatures by which all minimal solutions could be determined. On the other hand, if given x and b, the resolution of this problem means to find matrix $A = [a_{ij}]_{i \in I, j \in J}, 0 \le a_{ij} \le 1$ such that (2.13) holds. The minimum solution of (2.13) is given as $\breve{A} = x^{-1} \circ b$. Similarly, there is no well known method by which all maximal solutions could be determined.

2.4. Bipolar fuzzy relation equations

Bipolar FRE can be seen as extension of FRE involving the decision variable and its negation at the same time. There are some fields where, there is need for a variable which shows bipolar characteristics. So modification can be made in natural FRE. Let $A^+ = [a_{ij}^+],\ 0 \le a_{ij}^+ \le 1,\ A^- = [a_{ij}^-],\ 0 \le a_{ij}^- \le 1$ are $m \times n$ dimensional matrices and $b = [b_j],\ 0 \le b_j \le 1$ is a $n-$dimensional vector, then, the bipolar FRE with sup-$\otimes$ composition is given as

$$x \circ A^+ \vee \overline{x} \circ A^- = b \tag{2.15}$$

where $\circ$ denotes the sup-$\otimes$ composition, $\otimes$ being a continuous t-norm, i.e.

$$\max_{i \in I} [\max(x_i \otimes a_{ij}^+), \max(\overline{x}_i \otimes a_{ij}^-)] = b_j, \forall\ j \in J \tag{2.16}$$

The resolution problem of FRE (2.15) is to determine a vector $x = [x_i]_{i \in I}, 0 \le x_i \le 1$ such that (2.15) holds, where $\overline{x}_i = 1 - x_i, 0 \le \overline{x}_i \le 1$. The solution set of bipolar FRE whenever nonempty, can be characterised by a finite set of maximal and minimal solution pairs.

2.5. Fuzzy relation inequalities

FRE are special form of fuzzy relation inequalities. Let $A = [a_{ij}]_{m \times n},\ 0 \le a_{ij} \le 1,\ A' = [a'_{ik}]_{m \times s},$ $0 \le a'_{ik} \le 1,$ are fuzzy matrices, $b = [b_j]_{1 \times n}, 0 \le b_j \le 1, b' = [b'_k]_{1 \times s}, 0 \le b'_k \le 1$ are fuzzy vectors. System of fuzzy relation inequalities are defined as

$$x \circ A \ge b \text{ and } x \circ A' \le b' \tag{2.17}$$

where $\circ$ denotes the sup-$\otimes$ composition, $\otimes$ being a continuous t-norm, i.e.

$$\max_{i\in I}(x_i \otimes a_{ij}) \geq b_j \ \forall\ j \in J \ \text{ and } \max_{i\in I}(x_i \otimes a'_{ik}) \leq b'_k \ \forall\ k \in K \tag{2.18}$$

The resolution problem of (2.17) means to find a vector $x = [x_i]_{i\in I}, 0 \leq x_i \leq 1$ such that (2.17) holds. When solution set is non-empty it can be determined by one maximum solution and finite number of minimal solutions.

There are several other types of FRE discussed in literatures, with different compositions. Details of it are given in review of literature.

3. Benefits and applications of FRE

Fuzzy sets are main elements of fuzzy theory which is an extension of the classical theory. A classical set is defined by a two value function which only takes the values 0 and 1, but a fuzzy set is characterised by a membership function which takes all the intermediate values between 0 and 1. Therefore it is difficult to deal problems having linguistic information (vague information) using classical theory. The functional relation between input and output variables of systems with incomplete or linguistic kinds of information can be modelled best using fuzzy relation equations. FRE are powerful tools to analyze linear and nonlinear systems. Since solutions to a system of FRE may not be unique in general, solutions with some particular features are usually desired, for instance, Sanchez [52] suggested that the study of FRE can be extended to the study of the degree of fuzziness of the solutions. This immediately leads to the problem of optimizing an objective function (or multiple objective functions) subject to a system of FRE. Application of FRE lies in various areas, such as image processing, control problem, threat analysis, cause and effect analysis, medical diagnosis, etc.

Rotshtein [49] discussed a fault diagnosis problem based on a cause and effect analysis which is described by fuzzy relation equations. The input- output connections is done by interpreting the compositional Zadeh's rule of inference $B = A \circ R$ where B, R are known and A is unknown. Application of fuzzy relation equations in image processing can be found in Hirota et al. [15] and Loia et al. [34]. An image can be expressed by a fuzzy relation $S \in \Im(X \times Y)$ of $M \times N$ pixels by normalising the intensity of each pixel into [0, 1]. The image S, is compressed into a fuzzy relation $T \in \Im(I \times J)$ of $P \times Q$ pixels, where $P < M, Q < N$. Similarly, a compressed image expressed as a fuzzy relation $T \in \Im(I \times J)$; can be decompressed into a fuzzy relation $\tilde{S} \in \Im(X \times Y)$.

Applications of fuzzy relation equations in medical diagnosis can be found in Rotshtein and Rakityanskaya [48]. Consider universal set S of n symptoms, universal set C of m causes and universal set P of l patients. If A is a fuzzy relation between patients and symptoms, defined as $A \in \Im(P \times S)$ and if B is a fuzzy relation between patients and causes, defined as $B \in \Im(P \times C)$ then the fuzzy relation $R \in \Im(S \times C)$, between symptoms and causes can be represented as $B = A \circ R$, where $\circ$ denotes fuzzy composition. The fuzzy relation R gives the degree to which symptom S implies cause C. More work in this regard can be found on Sanchez [52].

4. Review of literature

Sanchez [51] first studied the problem of resolution of max-min FRE. Higashi and Klir [14] studied the general schemes for solving FRE with max-min composition and proposed an

algorithm to solve the problem. Stamou and Tzafestas [55] discussed the resolution of FRE with sup-$\otimes$ composition, $\otimes$ being an Archimedean t-norm. The solution existence problem was first studied and solvability criteria for composite FRE of any t norm were given. They gave the notion of mean solutions defined as the minimal element of the intersection of the lattices $[\breve{x}, \hat{x}]$. Obviously, the mean solution always exists since $\bar{x}$ always belongs to the intersection of the lattices $[\breve{x}, \hat{x}]$ and it is unique.

Luoh et al. [36] considered the problem of solving FRE with sup-$\otimes$ composition, $\otimes$ being a continuous t-norm. A computer based algorithm was proposed to solve the problem which operates systematically and graphically on a matrix pattern to get all the solutions of the problem. Markovskii [38] considered max-product FRE and showed that solving these equations is closely related with the covering problem, which belongs to the category of *NP*-hard problems. In terms of the covering problem, the conditions of compatibility of equations, of redundancy of equations, of uniqueness of solution, of uniqueness of minimal solutions were determined and it was shown that minimal solutions of equations correspond to irredundant coverings.

Li and Fang [23] showed that the systems of sup-T equations, T being a continuous triangular norm can be divided into two categories depending on the involved triangular norm. When the triangular norm is Archimedean, the minimal solutions correspond one-to-one to the irredundant coverings of a set covering problem. When it is non-Archimedean, they only correspond to a subset of constrained irredundant coverings of a set covering problem. It was also shown that the problem of minimizing a linear objective function subject to a system of sup-T equations can be reduced into a 0-1 integer programming problem in polynomial time.

Lin [27] considered the problem of solving sup-$\otimes$ composition FRE, $\otimes$ being an Archimedean t-norm. They provided a one-to-one correspondence between the minimal solutions of the equations and the irredundant coverings, as previously discovered by Markovskii [38] for FRE with max-product composition.

The sup-$\otimes$, $\otimes$ being a continuous t-norm is not the only possible composition between two fuzzy relations. In [51], Sanchez suggested the dual composition expressed by $\inf - \rightarrow$. The properties of this composition have been investigated in [3,43] where the necessary and sufficient condition of solvability and the characterization of a set of all solutions have been presented. However, the inf-$\rightarrow$ composition is not so popular in applications as its counterpart and therefore, only basic things regarding the solvability of systems of FRE are known.

Study of fuzzy linear optimization problems is an important ongoing topic of research. Since the feasible domain of FRE is, in general, non-convex, thus for linear optimization models with FRE as constraints, traditional/conventional methods fail. Linear optimization problems with FRE as constraints forms interesting topic of research due to scarcity of methods and fast algorithms for the resolution of FRE.

Fang and Li [5] in their maiden paper, considered an optimization model with a linear objective function subject to a system of max-min FRE. Feasible domain was first characterized and then the problem was converted to an equivalent problem involving 0-1 integer programming with a branch-and-bound solution technique. Loetamonphong and Fang [32] considered the linear optimization problem with max-product FRE composition by

deriving special characteristics of its feasible domain and the optimal solutions. Some procedures were proposed for reducing the original problem. The problem was transformed into a 0-1 integer program solvable by branch-and-bound method. Wu et al. [58] improved Fang and Li's method by proposing an initial upper bound for the branch-and-bound procedure and rearranging the data in parts so that the branches in the solution tree generated by branch-and-bound procedure are likely to be fathomed by the incumbent upper bound. Pandey [44] studied the optimization of FRE with continuous t-norms and with linear objective function. Pandey and Srivastava [45] gave efficient procedure for optimization of linear objective function subject to FRE as constraints and solved the associated 0-1 integer programming problem by branch-and-bound method with forward jumping-tracking technique. More work in this regard can be found in [7,8,17,47,59]. Thapar and Pandey [56] considered an optimization model with a linear objective function subject to max-t FRE as constraints, where t is an Archimedean t-norm. They applied the concept of covering problem to establish 0-1 integer programming problem equivalent to linear programming problem and proposed a binary coded genetic algorithm to obtain the optimal solution.

Ghodousian and Khorram [9] studied linear objective function optimization with respect to the fuzzy relational inequalities defined by max-min composition in which fuzzy inequality replaces ordinary inequality in the constraints. Fuzzy inequalities enable to attain the optimal points that are better solutions than those resulting from the resolution of the similar problems with ordinary inequality constraints. An algorithm was presented to generate such optimal solutions. More work in linear optimization problem with max-min, max-product and max-average fuzzy relation inequalities as constraints can be found in [12], [41] and [53] respectively. Molai [40] discussed a quadratic programming problem with max- product fuzzy relational inequalities. Yang et al. [63] proposed a max-min programming problem with addition-min fuzzy relation inequalities.

Nonlinear optimization model with a nonlinear objective function subject to a system of max-min FRE was first considered by Lu and Fang [35]. They designed genetic algorithm to be domain specific by taking advantage of the structure of the solution set of FRE. The individuals from the initial population were chosen from the feasible solution set and were kept within the feasible region during the mutation and crossover operations. More work on the same problem with max-min, max-average and max-t can be found in [24], [18] and [28] respectively.

Shivanian and Khorram [54] studied monomials geometric programming with fuzzy relation inequalities constraint with max-product composition. Simplification operations were given to accelerate the resolution of the problem by removing the components having no effect on the solution process. Wu [60] investigated the problem of minimizing an objective function with single-term exponents subject to FRE specified in max-min. Some properties for this optimization problem under the assumption of both negative and nonnegative exponents in the objective function were established and an efficient procedure to solve this optimization problem without looking for all the potential minimal solutions was proposed.

Loetamonphong et al. [33] studied class of optimization problems with multiple objective functions subject to a set of max-min FRE. Since the feasible domain of such a problem is in general non-convex and the objective functions are not necessarily linear, traditional optimization methods may become ineffective and inefficient. Therefore, taking advantage of the special structure of the solution set, they developed a reduction procedure to simplify the problem. They proposed a genetic-based algorithm to find the Pareto optimal solutions.

Thapar et al. [57] discussed satisficing solutions for multiobjective optimization problems with max-product FRE as constraints using genetic algorithm. Other significant literature on multiobjective optimization with FRE can be found in [13,19].

Wu et al. [61] considered a linear fractional programming problem with max-Archimedean FRE as constraints. They developed some theoretical results based on the properties of max-Archimedean t-norm and some procedures to reduce the feasible domain. The problem was simplified and converted into a traditional linear fractional programming problem. Chakraborty and Gupta [4] proposed a methodology for reducing the complexity of multiobjective linear fractional programming problem and to obtain efficient solution. Suitable transformations were applied to formulate an equivalent multiobjective linear programming problem. Using fuzzy set theoretic approach a procedure was explored. More work in this regard can be found in Khorram et al. [20].

Currently, the theory of FRE relies, by and large, on the assumption that the solution set is not empty. However, this is often not the case in practical applications. As the complexity of inverse problem grows, it becomes harder to find exact solutions. From a practical point of view, one might not be satisfied with a negative answer when no solution exists. A broader theory that would allow us to determine an adequate approximate solution is needed when no real solution exists. Therefore, various authors have proposed alternative methods to come up with approximate solutions: solutions maximizing some solution index or minimizing some distance function, solutions obtained by neural networks or by means of genetic algorithms. Existing methods for obtaining approximate solutions of FRE are reviewed by Di Nola et al. [42]. One of the methods, the first one described in the literature, is based upon a modified Newton method for solving FRE numerically. According to this method, which was proposed by Pedrycz [46], approximate solutions are those solutions x for which $x \otimes A$ has a minimal distance from b, $\otimes$ being the sup-t composition. Unfortunately, the structure of the approximate solution set obtained by this method was not clear. Another method, proposed by Gottwald and Pedrycz [11], is based on a solvability index, which, in turn, is based on the equality index introduced for fuzzy sets by Gottwald [10]; the larger the solvability index, the easier to solve the equation. According to the method, the given equation is appropriately modified to increase the solvability index until an equation is obtained that is solvable. But this method was rather inefficient and leads usually to a trivial solution. Klir and Yuan [21] generalized approximate solutions of FRE by employing the equality index proposed by Gottwald [10]. They introduced a goodness measure of the performance of approximate solutions and derived a lower bound and an upper bound of solvability of systems of FRE. More work on finding best approximate solutions of FRE can be found in [22,37,62].

Freson et al. [6] considered a generalization of the linear optimization problem subject to a system of bipolar FRE with max-min equations. The solution set of a system of bipolar fuzzy relational equations with max-min composition can be characterized by a finite set of maximal and minimal solution pairs. According to Liu et al. [29], determining the consistency of a system of bipolar fuzzy relational equations is NP-complete. Thus, the solution procedure for solving a system of bipolar fuzzy relational equations with max-min composition contains the high computation complexity. They proposed some properties to reduce the difficulty of solving the bipolar fuzzy relational equations with max-min composition and easily detect the case of an empty solution set. More work in this regard with max-product bipolar FRE can be found in Liu et al. [31].

Li and Jin [26] studied bipolar max-min FRE. A relation between the consistency of bipolar max-min equations and the classical boolean satisfiability problem was revealed. It was also shown that the problem of determining whether a system of bipolar max-min equations is consistent or not is NP-complete. And a consistent system of bipolar max-min equations, as well as its solution set, can be fully characterized by a system of integer linear inequalities.

Li and Liu [25] discussed the linear optimization problem with max-T bipolar FRE as constraints, T being a Łukasiewicz t-norm. They showed that this NP-hard problem can be reformulated in polynomial time into a 0-1 integer linear optimization problem. Liu et al. [30] considered a linear optimization problem with max-Łukasiewicz bipolar FRE and proposed a simple solution procedure to reduce the problem and finding the optimal solution without converting the problem to a 0-1 IPP. Aliannezhadi et al. [1] considered linear programming problem subject to the bipolar fuzzy relation equation constraints with the max-parametric hamacher composition. They presented some necessary and sufficient conditions for its solution existence. Problem was converted to an equivalent programming problem solvable by branch and bound method.

The problem of minimizing a nonlinear objective function with max-Łukasiewicz bipolar FRE can be found in Zhou et al. [64]. They reformulated the solution set as a system of bipolar fuzzy relational equations as a system of 0-1 mixed integer inequalities which can be handled within the framework of 0-1 mixed integer optimization. Aliannezhadi et al. [2] considered maximization of a monomial geometric programming problem subject to the bipolar max-product FRE and proposed some sufficient conditions for its solution existence and designed an algorithm to solve the problem. Mazraeh et al. [39] gave a genetic algorithm to solve a nonlinear objective function with max-min bipolar FRE, based on the structure of the feasible domain bipolar FRE.

Hu et al. [16] recently presented a multiobjective optimization problem constrained by a system of max-product bipolar FRE as constraints. An integer optimization based technique for order of preference by similarity to the ideal solution has been proposed by them for solving the problem.

Applications of FRE in different areas such as knowledge engineering, medical diagnosis, image processing, etc. can be found in [43,50,52].

5. Proposed work

There are several different types of FRE based on specific compositions. In the literature it has been observed that the methods and solution set of different types of FRE changes according to the compositions. Moreover, necessary and sufficient conditions for solvability vary among different types of FRE. Thus resolution problem of different types of FRE is one of the most broadly studied problems in the area of fuzzy sets and systems. As there is limitation to finding all minimal /maximal solutions of system of FRE, so we can associate optimization problem to it. Optimization models subject to system of FRE is broadly studied and an important topic of research. Linear and nonlinear systems can be best analyzed by system of FRE. Since the solution set of FRE is, in general non-convex, hence we cannot apply traditional methods to solve linear and nonlinear optimization problems subject to system of FRE.

The major objective behind the work lies in the study of different types of FRE and linear, nonlinear and multiobjective optimization problems with FRE as constraints. The two major types of FRE are with sup-$\otimes$ composition and inf-$\rightarrow$ composition, where $\otimes$ and $\rightarrow$ denotes a t-norm and a residuation operation (implication), respectively. The general model of optimization problems we will consider are:

$$\text{Optimize } \{f_k(x)\}_{k\in K},\ K=\{1,2,\ldots,p\} \tag{5.1}$$

$$\text{s.t. } x \circ A = b \tag{5.2}$$

$$x \in [0,1]^m$$

where $A=[a_{ij}]_{m\times n},\ 0\le a_{ij}\le 1,$ is system representing $m\times n$ dimensional fuzzy relation matrix, $b=[b_j]_{1\times n},\ 0\le b_j\le 1,$ is $1\times n$ dimensional matrix, $x=[x_i]$ is $1\times m$ dimensional decision matrix and $\circ$ denotes sup-$\otimes$ composition of x and A, $\otimes$ being a continuous t-norm. For problem (5.1)-(5.2), the following cases arise:

(i) If $|K|=1$ and $f_k(x)$ is a linear function, then problem (5.1)-(5.2) is a linear optimization problem with sup-$\otimes$ fuzzy relation equations as constraints,

(ii) If $|K|=1$ and $f_k(x)$ is a nonlinear function, then problem (5.1)-(5.2) is a nonlinear optimization problem with sup-$\otimes$ fuzzy relation equations as constraints,

(iii) If $|K|>1$ where $f_k(x)$ is a linear function, nonlinear function or both, then problem (5.1)-(5.2) is a multiobjective optimization problem with sup-$\otimes$ fuzzy relation equations as constraints.

Similarly, optimization problems subject to other extensions of FRE such as bipolar FRE, fuzzy relation inequalities, mixed FRE are to be explored along with methods for solving them.

Mentioned below are some research directions we intend to consider:

1. Exposition of FRE, choice of fuzzy compositions with suitable t-norms, characterization of the feasible domain, complete solution set and establishing necessary and sufficient conditions for solvability of FRE,

2. Designing of optimization models (linear models, nonlinear models & multiobjective models) with FRE as constraints, characterizing of models for optimal solutions/converging solutions/satisficing solutions,

3. Developing techniques, tools and algorithms based upon uncoded techniques as well as coded techniques. Coding techniques require the coding (representation) of variables in specific number system. In uncoding techniques, variables are usable in their original forms,

4. Is a given system of FRE consistent? If a given system of FRE is consistent, then how to characterize its solution set? If a given system of FRE is inconsistent, then how to resolve the inconsistency?

5. Finding approximate solutions of FRE having no unique solution,

6. Application of fuzzy relational models in real life instances.

Moreover, future research directions in this area can be optimization models with intuitionistic FRE as constraints. No significant literature in this direction has been found, but it is possible to consider linear, nonlinear and multiobjective decision models involving intuitionistic FRE with different compositions, by first establishing necessary and sufficient conditions for solvability of intuitionistic FRE.

Methodologies used to carry out the work are as follows:

(i) Uncoded techniques:
- Fuzzy relational calculus
- Modified classical techniques

(ii) Coded techniques (soft computing techniques):
- Genetic algorithms
- Evolutionary theory
- Graphical algorithms
- Hybrid techniques

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